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THE UPPER CERTIFIED DOMINATION NUMBER OF A GRAPH

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Abstract
A certified dominating set D of vertices in a connected graph G is minimal certified dominating set if no proper subset of D is an certified dominating set of G.The upper certified domination number $\Gamma_{cer}(G)$ is the maximum cardinality of a minimal certified dominating set of G. It is shown that for every positive integers a and b with $2 \leq a \leq b$ there exists a connected graph G suchthat $\gamma_{cer}(G) = a$ and $\Gamma_{cer}(G) = b$.

Keywords: upper certified domination number, certified domination number, domination number.
AMS Subject Classification: 05C69

1. Introduction:

Let $G = (V, E)$ be a finite, undirected graph without loops and multiple edges. Unless and otherwise stated, the graph $G = (V, E)$ has $n = |D|$ vertices and $m = |E|$ edges. For basic definitions and terminologies, we refer to[4].Two vertices u and v are said to be adjacent if uv is an edge of G.The open neighbourhood of a vertex v in a graph G is defined as the set $N_G(v) = \{u \in V(G): uv \in E(G)\}$. While the closed neighbourhood of a vertex v in a graph G is defined as the set $N_G[v] = N_G(v) \cup \{v\}$.

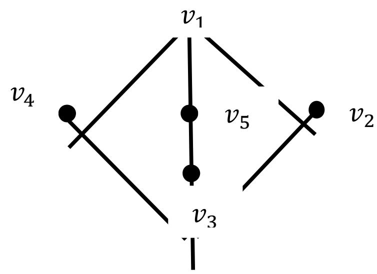
A set $D \subseteq V(G)$ is called a dominating set of G if for every $v \in V \backslash D$ is adjacent to atleast one vertex in D. A dominating set D is said to be minimal if no subset of D is a dominating set of G. The minimum cardinality of a minimal dominating set of G is called the domination number of G and is denoted by $\gamma(G)$. Any dominating set of cardinality $\gamma(G)$ is a γ-set of G. Recently dominating number of a graph is studied in [4]. A dominating set D of $G = (V, E)$ is a certified dominating set, if every vertex in D has either zero or atleast two neighbours in $V \backslash D$.The certified domination number $\gamma_{cer}(G)$ of G is the minimum cardinality of certified dominating set. The certified domination number of a graph was studied in [5,6]. In this article, the upper certified domination number of a graph is introduced and studied.

2. The Upper Certified Domination Number of a Graph

Definition 2.1

A certified dominating set D of vertices in a connected graph G is minimal certified dominating set if no proper subset of D is an certified dominating set of G. The upper certified domination number $\Gamma_{cer}(G)$ is the maximum cardinality of a minimal certified dominating set of G

Example 2.2



For the graph G given in Figure 2.1, $D_1 = \{v_1, v_3\}$ and $D_2 = \{v_2, v_4, v_5\}$ is the minimal certified dominating set. Therefore $\gamma_{cer}(G) = 2$, $\Gamma_{cer}(G) = 3$.

The following theorem follow from the definition of the upper certified domination number of a graph

Theorem 2.3. For the cycle $G = C_n (n \geq 4)$,$\Gamma_{cer}(G) = \begin{cases} \frac{n}{2} & if \ n \ is \ even \\ \frac{n-1}{2} & if \ n \ is \ odd \end{cases}$

Proof: Let $V(G) = \{v_1, v_2, \ldots, v_n\}$

We have the following cases

Case (i) n is even. Let $n = 2k, k \geq 2$

Let $D = \{v_2, v_4, v_6, \ldots, v_{2k}\}$.Then n is a minimal certified dominating set of G and so $\Gamma_{cer}(G) \geq k = \frac{n}{2}$. We prove that so $\Gamma_{cer}(G) = k$. On the contrary suppose that $\Gamma_{cer}(G) \geq k + 1$. Then there exists a minimal certified dominating set D' of G suchthat $|D'| \geq k + 1$ then there exists $x \in D'$ such that $x \notin D$. Then there exists $y, z \in D'$ suchthat $G[x, y, z]$ is a path in C_n. Let $D_1 = D' - \{y\}$. Then $D_1$is a certified dominating set of G such that $D_1 \subset D'$, which is a contradiction to D' a minimal certified dominating set of G.Therefore $\Gamma_{cer}(G) = k = \frac{n}{2}$

Case (ii) n is odd.

For $= 5$, $D_1 = \{v_1, v_4\}$,$D_2 = \{v_1, v_3\}$,$D_3 = \{v_2, v_5\}$,$D_4 = \{v_2, v_5\}$ and $D_5 = \{v_3, v_5\}$ are the only five minimal certified dominating set of G so that $\Gamma_{cer}(G) = 2$. So let $n \geq 7$, Let $D = \{v_1, v_3 , v_5, \ldots, v_{2k-1}\}$. Then D is minimal certified dominating set of G and so $\Gamma_{cer}(G) \geq k = \frac{n-1}{2}$. We prove that $\Gamma_{cer}(G) = k$.On the contrary, suppose that $\Gamma_{cer}(G) = k + 1$.Then there exists a minimal certified dominating set D' of G such that $|D'| \geq k + 1$.Then there exists $x \in D'$ such that $x \notin D$. Let $y \in D'$ such that $xy \in E(G)$.Then $D_1 = D' - \{x\}$ is a certified dominating set of G with $D_1 \subset D'$, which is a contradiction to D' a minimal certified dominating set of G,which is a contradiction. Therefore $\Gamma_{cer}(G) = k = \frac{n-1}{2}$.

Theorem 2.4. For the path $G = C_n (n \geq 4)$, $\lceil_{cer}(G) = \begin{cases} 4 & if\ n = 4 \\ 2 & if\ n = 5 \\ 3 & if\ \ n = 7 \\ \frac{n}{3} & if\ n \equiv 0(mod3) \\ \frac{n-7}{3}\ if\ n \equiv 1(mod3) and\ n \geq 7 \\ \frac{n-5}{3}\ if\ n \equiv 2(mod3) and\ n \geq 8 \end{cases}$

Proof: Let P_n be $v_1, v_2, \ldots, v_n$.

Case (i) $n = 4$. Let $D = \{v_1, v_2, v_3, v_4\}$ is the unique minimal certified dominating set of G so that $\lceil_{cer}(G) = 4$.

Case (ii) $n = 5$. Let $D = \{v_2, v_4\}$. Then D is the unique minimal certified dominating set of G so that $\lceil_{cer}(G) = 2$.

Case (iii) $n = 7$. Let $D = \{v_2, v_4, v_6\}$. Then D is the unique minimal certified dominating set of G so that $\lceil_{cer}(G) = 3$.

Case (iv) $n \equiv 0(mod\ 3)$,$n = 3k, k \geq 1$. Let $D = \{v_3, v_6, v_9, \ldots, v_{3k}\}$ is the minimal certified dominating set of G so that $\lceil_{cer}(G) \geq \frac{3k-3}{3} + 1 = k = \frac{n}{3}$ .We prove that $\lceil_{cer}(G) = \frac{n}{3}$.On the contrary suppose that $\lceil_{cer}(G) \geq \frac{n}{3} + 1$.Then there exists a minimal certified dominating set S' of G suchthat $|D'| \geq \frac{n}{3} + 1$ then there exists $x \in D'$ such that $x \notin D$.Let $y \in D'$ such that $xy \in E(G)$.Then $D_1 = D' - \{x\}$ is a certified dominating set of G with $D_1 \subset D'$ which is a contradiction to D' a minimal certified dominating set of G, which is a contradiction. Therefore $\lceil_{cer}(G) = k = \frac{n}{3}$.

Case (v) $n \equiv 1(mod\ 3)$,$n = 3k + 1, k \geq 2$. Then $D = \{v_{10}, v_{13}, v_{16}, \ldots, v_{3k+1}\}$ is the minimal certified dominating set of G so that $\lceil_{cer}(G) \geq \frac{3k+1-10}{3} + 1 = k - 2 = \frac{n-7}{3}$ .We prove that $\lceil_{cer}(G) = \frac{n-7}{3}$. On the contrary suppose that $\lceil_{cer}(G) \geq \frac{n-7}{3} + 1$.Then there exists a minimal certified dominating set D' of G suchthat $|D'| \geq \frac{n-7}{3} + 1$ then there exists $x \in D'$ such that $x \notin D$. Let $y \in D'$ suchthat $xy \in E(G)$.Then $D_1 = D' - \{x\}$ is a certified dominating set of G with $D_1 \subset D'$ which is a contradiction to D' a minimal certified dominating set of G, which is a contradiction. Therefore $\lceil_{ce}\ (G) = k = \frac{n-7}{3}$.

Case (vi) $n \equiv 2(mod\ 3)$,$n = 3k + 2, k \geq 2$. Then $D = \{v_8, v_{11}, v_{14}, \ldots, v_{3k+2}\}$ is the minimal certified dominating set of G so that $\lceil_{cer}(G) \geq \frac{3k+2-8}{3} + 1 = k - 1 = \frac{n-2}{3} - 1 = \frac{n-5}{3}$. Similarly by case (v), we prove that $\lceil_{cer}(G) = k = \frac{n-5}{3}$.

Theorem 2.5. For the complete graph $G = K_n$, $\lceil_{cer}(G) = n$

Proof: Since every vertex of the complete graph, $K_n (n \geq 4)$ is an extreme vertex, the vertex set of K_n is the unique certified dominating set of K_n. Thus $\lceil_{cer}(G) = n$.

Theorem 2.6. For the complete bipartite graph $= k_{r,s}$, $\lceil_{cer}(G) = S$.

Proof: Consider $k_{r,s}$ with V($k_{r,s}$)= $\{u_i, v_j / 1 \leq i \leq r, 1 \leq j \leq s\}$ with partition $V_1 = \{v_1, v_2, \ldots, v_r\}$ and $V_2 = \{u_1, u_2, \ldots, u_s\}$. Let $S = V_2$ is a minimal certified dominating set of G so that $\lceil_{cer}(G) \geq s$. we prove that $\lceil_{cer}(G) = S$. On the contrary suppose that $\lceil_{cer}(G) \geq s + 1$.Then there exists a minimal certified dominating set S' of G suchthat $|S'| \geq S + 1$. Then

$S' \subset V_1 \cup V_2$ then there exists $x \in V_1$and $y \in V_2$ suchthat $S_1 = \{x, y\}$ is a certified dominating set of G with $S_1 \subset S'$ which is a contradiction. Therefore $\lceil_{cer}(G) = S$.

Theorem 2.7. For the helm graph $= H_n$, $\lceil_{cer}(G) = n$.

Proof: Let $S = \{v_1, v_2, \dots, v_n\}$ be the set of cut vertices of G.Then the set S is the unique certified dominating set of G so that $\lceil_{cer}(G) = n$.

Theorem 2.8. For the barbell graph $G = B_n(n \geq 3)$, $\lceil_{cer}(G) = 2$.

Proof: Let G be a barbell graph which is obtained by joining two complete graph by a bridge. Let $e = \{v_1u_1\}$ be the edge joining the two complete graph. Let the vertices of the two complete graph be $\{v_i\}(1 \leq i \leq n)$ and $\{u_i\}$ $(1 \leq j \leq m)$.Let $V(G) = \{v_1, v_2, \dots, v_n, u_1, u_2, \dots, u_m\}$ be the vertices of G.Since v_1 is adjacent to $\{v_2, v_3, \dots, v_n\}$ and u_1 is adjacent to $\{u_2, u_3, \dots, u_m\}$ where $deg(v_1) = n$ and $deg(u_1) = m$. Let $D = \{v_1, u_1\}$.Then D is a minimal certified dominating set of G and so $\lceil_{cer}(G) \geq 2$. We prove that $\lceil_{cer}(G) = 2$ .On the contrary suppose that $\lceil_{cer}(G) \geq 3$.Then there exists a minimal certified dominating set D' of G suchthat $|D'| \geq 3$ then there exists $x \in D'$ such that $x \notin D$.Then $D_1 = D' - \{x\}$ is a certified dominating set of G with $D_1 \subset D'$ which is a contradiction to D' a minimal certified dominating set of G, which is a contradiction. Therefore $\lceil_{cer}(G) = 2$.

Theorem 2.9. For the windmill graph $G = W_{m,n}$ $(m \geq 3, n \geq 2)$ $\lceil_{cer}(G) = a$.

Proof: Let G be a windmill graph $W_{m,n}$ is the graph obtained by taking m copies of the complete graph K_n with a vertex in common. Let $D = \{v_1\}$ is a certified dominating set of G.Then the set $D' = \{u_1, u_2, \dots, u_a, v_1, v_2, \dots, v_a\}$ is the minimal certified dominating set of G so that $\lceil_{cer}(G) \geq a$. We prove that $\lceil_{cer}(G) = a$. On the contrary suppose that $\lceil_{cer}(G) \geq a + 1$.Then there exists a minimal certified dominating set D' of G suchthat $|D'| \geq a + 1$ then there exists $x \in D'$ such that $x \notin D$.Let $y \in D'$ suchthat $xy \in E(G)$.Then $D_1 = D' - \{x\}$ is a certified dominating set of G with $D_1 \subset D'$ which is a contradiction to D' a minimal certified dominating set of G, which is a contradiction. Therefore $\lceil_{cer}(G) = a$.

Theorem 2.10. For every positive integers a and b with $2 \leq a \leq b$ there exists a connected graph G suchthat $\gamma_{cer}(G) = a$ and $\lceil_{cer}(G) = b$.

Proof: Let $P_i: x_i, y_i$ $(1 \leq i \leq b - a + 1)$ be a copy of path on two vertices and $Q_i: u_i, v_i, w_i$ $(1 \leq i \leq a - 1)$ be a copy of path on three vertices. Let G be the graph obtained from $P_i(1 \leq i \leq b - a + 1)$ and Q_i $(1 \leq i \leq a - 1)$ by adding the new vertex x and introducing the edges $xv_i(1 \leq i \leq a - 1)$, xx_i and xy_i $(1 \leq i \leq b - a + 1)$.The graph G is given in figure 2.2. First we prove that $\gamma_{cer}(G) = a$. It is easily observed that every γ_{cer}-set of G contains each $v_i(1 \leq i \leq a - 1)$ and the vertex x and so $\gamma_{cer}(G) \geq a$. Let $S = \{x, v_1, v_2, \dots, v_{a-1}\}$. Then S is a γ_{cer}-set of G so that $\gamma_{cer}(G) = a$.

Next we prove that $\lceil_{cer}(G) = b$. Let $D = \{v_1, v_2, \dots, v_{a-1}\} \cup \{x_1, x_2, \dots, x_{b-a+1}\}$. Then D is a certified dominating set of G. We prove that D is a minimal certified dominating set of G. On the contrary suppose that D is not a minimal certified dominating set of G then there exists a certified dominating set D' of G suchthat $D' \subset D$. Then there exists $u \in D'$ such that $u \notin D$. If $u = v_i(1 \leq i \leq a - 1)$, then u_i, w_i is not dominated by any element of D'. If $u = x_i$ for some $i(1 \leq i \leq b - a + 1)$, then $y_i(1 \leq i \leq b - a + 1)$ is not dominated by every element of D', which is a contradiction to D' a certified dominating set of G. Therefore D is a minimal certified dominating set of G and so $\lceil_{cer}(G) \geq b$. We prove that $\lceil_{cer}(G) = b$. On the contrary suppose that $\lceil_{cer}(G) \geq b+1$. Then there exist a minimal certified dominating set S' of G such that $|S'| \geq b+1$. Then $S \not\subset S'$ and $D \not\subset S'$. Then u_i and $w_i \in S'$ for all $i(1 \leq i \leq a-1)$. Hence it follows that S' is not a certified dominating set of G, Which is a contradiction. Therefore $\lceil_{cer}(G) = b$.

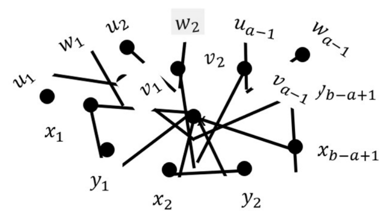


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Figure 2.2

Conclusion

In this article, the upper certified domination number of graphs is introduced and studied. This study can be extended to other certified domination parameter.

References

[1] V. G. Bhagavathi Ammal and A. Janani, Total certified Domination Number of a Graph, Journal of Applied Science and computations, 66(5), 2020, 90-98.

[2] V. G. Bhagavathi Ammal, R. Louisa Dickfania, Accurate certified domination number of graphs, International Journal of Mathematics Trends and Technology (IJMTT), 66(5), 2020, 90-98.

[7] E. J. Cochaye, R. M. Dawes., S. T. Hedetnieni., Total domination in graphs. Networks 10,211-219 (1980).

[3] M. Dettlaff, M. Lemanska, and Topp, Certified domination, AKCE International Journal of Graphs and Combinatorics, 17(1), (2020), 86-97.

[8] O. Favaron, M.A. Henning, C. Mynhart, J. Puech, Total domination in graphs with minimum degree three.J. Graph Theory 34, 9-19(2000).

[4] T. W. Haynes, S.T. Hedetniemi. and P. J. Slater, Fundamentals of Domination in Graphs, Marcel Dekker, New York, (1998).

[9] M. A. Henning, Survey of selected recent results on total domination in graphs, Discret.Math.309,32-63(2009).

[5] J. John and Malchijah, MS 2021, The forcing non-split domination number of a graph, Korean journal of mathematics, 29(1),1-12.

[6] J. John and V. Sujin Flower, On the forcing domination and the forcing total domination number of a graph, Graphs and Combinatorics, 38(2022), article number :142.

[15] P. C. B. Lam. B. Wei, On the total domination number of graphs, Utilitas Mathematica 72,223-240(2007).

[16] P. Zhang, the upper forcing geodetic number of a graph, Ars Comb.62,3-15(2012).